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A lower bound for the cyclic cutwidth of the n-cube

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A LOWER BOUND FOR THE CYCLIC CUTWIDTH OF THE N-CUBE

A Project

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

James Shigeo Namekata

June 1999

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James Shigeo Namekata

June 1999

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ABSTRACT

The linear cutwidth of a particular family of subgraphs of the n-dimensional cube, Q_n, is used to provide a good lower bound for the cyclic cutwidth of Q_n. A summary of previous research dealing with the cyclic cutwidth of graphs is investigated. A theorem stating that the linear cutwidth of a graph is equal to the cyclic cutwidth of the disjoint union of two copies of the same graph is proved. Next, a recurrence relation is solved and used to find the linear cutwidth of our subgraph of Q_n. By using the theorem, we are able to calculate the cyclic cutwidth of the union of two disjoint copies of our subgraph; thus, providing a lower bound for the cyclic cutwidth of the n-dimensional cube. Finally, this lower bound is improved by applying the theorem to $Q_{n-1} \cup Q_{n-1}$.

ACKNOWLEDGMENTS

I would like to thank my advisor, Dr. Joseph Chavez, whose patience and time was invaluable to me. His work and guidance gave me the inspiration to complete this project. I would also like to thank my family and friends for the support they gave me during the long hours of research that went into completing this project.

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1 INTRODUCTION

Over the last few years, a variety of graph labeling problems have sparked the interest of mathematicians. The bandwidth, edge sum, and cutwidth problems are problems in which one must label, or number, the vertices of a given graph in order to optimize some parameter. The particular problem we will study in this paper is the cutwidth problem, which basically involves finding a labeling of the vertices of a graph so that the maximum number of edges between consecutive vertices is minimized.

A graph, G, is defined as a structure with a set of vertices, V, and a set of edges, E, which join pairs of distinct vertices. An n-cube is an n-dimensional unit cube with 2^n vertices, each with degree n, and $n(2)^{n-1}$ edges. The vertices of an n-cube may be represented as n-tuples of 0's and 1's. Edges connect pairs of vertices that differ in only one coordinate in their n-tuples. We will denote the cube with 2^n vertices as Q_n.

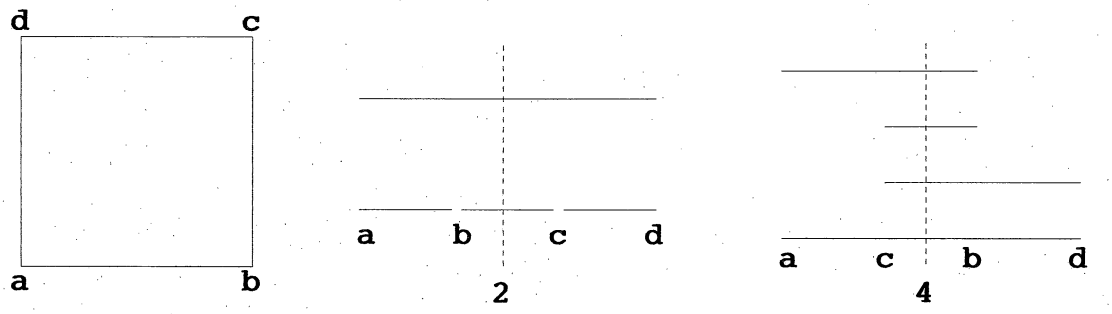
A vertex numbering of G is a function:

$$\eta : V \rightarrow \{1,2,...,M\}, \text{ where } M = |V|,$$

which is one-to-one and onto. A numbering may be thought of as embedding the vertices of G onto a linear host graph (path). The cutwidth of G, with respect to η, $cw(G : \eta)$, is the maximum number of edges which pass between any two neighboring vertices on the host graph, H. Figure 1(a) depicts the graph of Q_2, which we label as G. Figure 1(b) shows one particular numbering (η_1) of the vertices of G onto the host graph, H, where the linear cutwidth is 2. Figure 1(c) shows another numbering (η_2) of the vertices where the linear cutwidth is 4. Then, the cutwidth of G

is:

$$cw(G) = \min_{\eta} \max_{\epsilon} |\{(v,w) \in E : \eta(v) \leq \epsilon \leq \eta(w)\}|.$$



G η_1 η_2

(a) (b) (c)

Figure 1

Thus, the cutwidth of the graph is the minimum cutwidth of all possible labelings of the vertices of G onto H. The cutwidth problem is to find the minimum cutwidth.

Neighboring vertices are vertices that are drawn next to each other on the host graph. Adjacent vertices are vertices that are connected by an edge. In Figure 2, vertex 'a' and vertex 'b' are neighboring vertices, but not adjacent vertices. Vertex 'b' and vertex 'd' are adjacent vertices, but not neighboring vertices. Vertex 'b' and vertex 'c' are both adjacent and neighboring vertices.

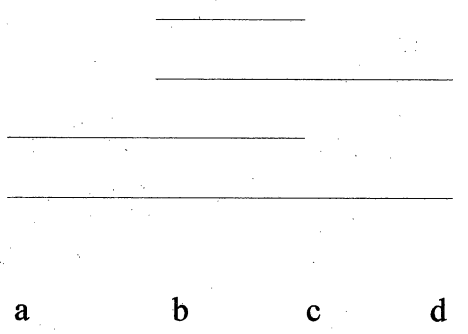


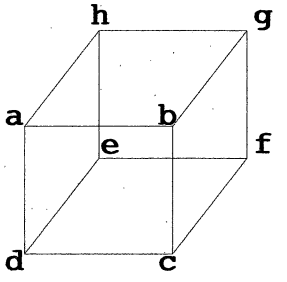
Figure 2

There are some special properties of an embedding of G onto H. If any two vertices are connected by an edge in G, then a path must connect the same two vertices on H. If a path travels past a vertex on the host graph, H, to get to a more distant vertex, then it is considered to pass through each vertex that it passes by. Thus, the cutwidth is the maximum number of paths passing between any two neighboring vertices of H. By changing the labeling of the vertices of G, we may change the cutwidth. If G has n vertices, then there are $n!$ embeddings of the vertices of G onto H. The solution of the linear cutwidth problem is the minimum of the cutwidths for each of the possible embeddings of G onto H. When the host graph is a path, the cutwidth problem is called the linear cutwidth of G, and it will be denoted by $lcw(G)$. When the host graph is a cycle, the cutwidth problem is called the cyclic cutwidth of G, and it will be denoted by $ccw(G)$.

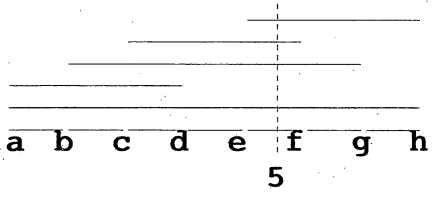
We may think of the cyclic host graph for an n-cube to be in the form of a 2^n-gon. The cyclic cutwidth of G, with respect to η, denoted $ccw(G : \eta)$, is the maximum number of paths that cut through a radius of H [6]. Then the cyclic cutwidth of G,

denoted $ccw(G)$, is the minimum value of $ccw(G : \eta)$ over all numberings η.

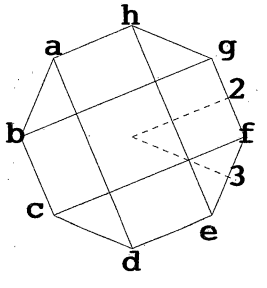
A simple example to examine is the three-dimensional cube, Q_3. See Figure 3a. A linear layout consists of the eight vertices of Q_3 embedded on a path, H. Let η_1 be the vertex numbering of the eight vertices of Q_3 shown in Figure 3b. The twelve edges which connect vertices in Q_3 must also connect the same vertices in H. The linear cutwidth is the maximum number of paths between any two neighboring vertices of H. So, in this example, $lcw(Q_3 : \eta_1) = 5$. The $lcw(Q_3)$ would be the minimum of the cutwidths for each of the possible embeddings of Q_3 onto H.



(a)



(b)



(c)

Figure 3

By comparison, we consider one numbering of the eight vertices of a three-dimensional cube on a cyclic host. Since there are eight vertices, the host graph, H, will be in the form of a regular octagon. The twelve edges of G must also be drawn as paths in H. Paths actually travel along the perimeter of the octagon passing through each vertex they pass by. Each path has two ways to travel around the octagon. Actually, so we can visualize the embedding, we will draw paths which connect the vertices with straight lines through the interior of the octagon.

Let η_2 be the numbering shown in Figure 3c. To calculate the cyclic cutwidth, $ccw(G : \eta_2)$, we construct a ray from the center of the octagon to the regions between each pair of neighboring vertices. The number of edges crossed is found for each pair of neighboring vertices and the maximum number of edges is recorded as the cutwidth. Thus, $ccw(Q_3 : \eta_2) = 3$. The $ccw(Q_3)$ would be the minimum of the cutwidths for each of the possible embeddings of Q_3 onto H.

2 SUMMARY OF THE LITERATURE

There have been countless hours of research spent trying to find the cyclic cutwidth of the n-dimensional cube. Many researchers as well as students have tried to solve this complex problem. The cyclic cutwidth problem is very difficult in general, but some results are known for special cases.

The survey paper by F. R. K. Chung [3] provides valuable background information about the cutwidth problem. Several results comparing the cutwidth of a complete k-level t-ary tree with its topological bandwidth are presented. But more

important to the present project are Chung's suggestions for future research of cutwidth problems. Most of the results examined by Chung concentrate on the case where the host graph is a path. Chung suggests finding the cutwidth for a variety of different host graphs. Some examples of other possible host graphs are grids, trees and cycles.

The cutwidth problem of the n-cube has been studied when the host graph is both a path and rectangular grid. The cutwidth of a graph when the host graph is a grid is called the congestion of the graph. A team of researchers including S. Bezrukov, M. Rottger, U-P Schroeder, L. Harper, and J. Chavez obtained a recurrence relation for the linear cutwidth of the n-dimensional cube [1]. By solving this recurrence, they found the linear cutwidth to be

$$lcw(Q_n) = \begin{cases} \frac{2^{n+1}-2}{3} & \text{if n is even} \\ \frac{2^{n+1}-1}{3} & \text{if n is odd.} \end{cases}$$

Following is their interesting result relating the congestion to the linear cutwidth of the n-cube.

Theorem (2.1) *$con(Q_n : P_{2^{n1}} \times P_{2^{n2}}) = lcw(Q_{n_2})$, where $n_1 + n_2 = n$, $n_2 \geq n_1$, and P_m is a path of m vertices.*

Thus, from this theorem, the linear cutwidth of an n-cube can be used to find its congestion.

Y. Lin [7] studied the cyclic cutwidth for the complete bipartite graph. Lin states that given any complete bipartite graph, $K_{m,n}$, with (X, Y) as the bipartition of $K_{m,n}$,

where $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$, an upper and lower bound can be calculated.

Theorem (2.2 [Lin]) *For any integers m, n, the following holds*:

$$\left\lceil \frac{1}{2}\left(\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil\right) \right\rceil \leq ccw(K_{m,n}) \leq \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil.$$

Lin also has a corollary that if m and n are both even or if $m = n$, then the cyclic cutwidth is given by the above lower bound.

Corollary (2.3 [Lin]) *If both m and n are even or m = n, we have*:

$$ccw(K_{m,n}) = \left\lceil \frac{1}{2}\left(\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil\right) \right\rceil.$$

Thus, we have both an upper and lower bound as well as exact results for special cases of the cyclic cutwidth of complete bipartite graphs.

In this paper, we are interested in finding a lower bound for the cyclic cutwidth of the n-dimensional cube. Although the exact solution has yet to be proven, many researchers have ideas of what the cyclic cutwidth of the n-cube should be. J. Chavez and R. Trapp [2] have conjectured that the n-cube's cyclic cutwidth can be calculated using the following formula.

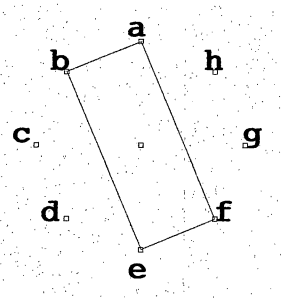
Conjecture (2.4 [Chavez-Trapp]) $ccw(Q_n) = \begin{cases} \frac{5(2^{n-2})-2}{3} & \text{if } n \text{ is even} \\ \frac{5(2^{n-2})-1}{3} & \text{if } n \text{ is odd.} \end{cases}$

This conjecture has been verified by computer for the three-dimensional cube. Ray Gregory [4] submitted his Masters project in which he used a computer program

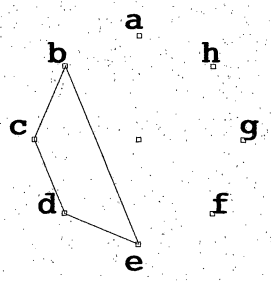
written in C++ code to calculate the cyclic cutwidth of the 3-cube. First, he found the number of ways to embed the eight vertices of the 3-cube onto an octagon. His computer program found and evaluated the cyclic cutwidth of all (5040)(4096) or 20,643,840 cases possible.

Beatrice James [6] verified the Chavez-Trapp conjecture for the 3-dimensional and 4-dimensional cubes. Her method of proving the cyclic cutwidth for the 3 and 4-dimensional cubes is interesting and unique. An outline of her proof for the 3-dimensional cube follows (the 4-dimensional case is similar).

James noticed that the 3-cube is composed of two disjoint 2-cubes connected by four distinct edges. She also noticed that when the vertices of the 3-cube were embedded onto the cyclic host, the two distinct 2-cubes can either contain the center or not contain the center. (See Figure 4.) Therefore, this particular proof had three cases to examine: both 2-cubes containing the center, one containing the center, and neither containing the center. And in each case, she found that the cyclic cutwidth had to be at least three; thus, proving that the cyclic cutwidth of the 3-cube is three.



Containing the center



Not containing the center

Figure 4

Francisco Rios [8], assisted by his mentor, R. Trapp, discovered some interesting relationships between the linear cutwidth and the cyclic cutwidth of complete graphs. First, a theorem for finding the linear cutwidth of any complete graph was proved. Next, he found a relationship between the linear cutwidth and the cyclic cutwidth of complete graphs based on the proofs of his previous two theorems. But the main point of interest in Rios' paper is his theorem for finding the cyclic cutwidth of complete graphs.

Theorem (2.5 [Rios]) *For any complete graph K_n on n vertices,*

$$ccw(K_n) = \begin{cases} \frac{n^2+8}{8} \text{ if } \frac{n}{2} \text{ is even} \\ \frac{n^2+4}{8} \text{ if } \frac{n}{2} \text{ is odd} \\ \frac{n^2-1}{8} \text{ if } n \text{ is odd}. \end{cases}$$

Since the n-dimensional cube is a subgraph of the complete graph with 2^n vertices, then the results of the cyclic cutwidth of complete graphs provides us with a result which will be more than the cyclic cutwidth of the n-dimensional cube. But an interesting relationship between the linear cutwidth and cyclic cutwidth is his corollary to his theorem. This corollary will be of interest to this paper.

Corollary (2.6 [Rios]) *For any complete graph K_n on n vertices,*

$$ccw(K_n) = \begin{cases} \frac{lcw(K_n)+2}{2} \text{ if } \frac{n}{2} \text{ is even} \\ \frac{lcw(K_n)+1}{2} \text{ if } \frac{n}{2} \text{ is odd} \\ \frac{lcw(K_n)}{2} \text{ if } n \text{ is odd}. \end{cases}$$

By Rios' corollary, the savings of cutwidth from using a cyclic host compared to the linear host is half.

At the other end of the spectrum, J. Chavez and R. Trapp [7] found an important relationship between the linear cutwidth and the cyclic cutwidth of trees, which shows that no savings is achieved.

Theorem (2.7 [Chavez-Trapp]) *If T is a tree, then $lcw(T) = ccw(T)$.*

This relationship states that given any tree, the linear cutwidth and the cyclic cutwidth are equal. We know that the linear cutwidth is always greater than or equal to the cyclic cutwidth since whenever a linear chassis is embedded onto a cycle, some of the edges may wrap around the cycle; thus, reducing the cut. But if G does not contain any cycles, as is the case for a tree, then we expect the linear cutwidth and the cyclic cutwidth to be equal.

In order to prove that the linear and cyclic cutwidths are equal, Chavez and Trapp proved that the linear cutwidth is less than or equal to the cyclic cutwidth. They did this by describing an algorithm which produces a linear layout of a tree, T, from a cyclic layout without increasing the cutwidth; thus, proving the linear cutwidth of a tree is equal to the cyclic cutwidth of a tree.

One approach toward solving the cyclic cutwidth of the n-dimensional cube is to find an upper and lower bound. Lin and Chung both used upper and lower bounds to narrow down their results. This technique of using bounds is not uncommon in combinatorics. An established upper bound for the cyclic cutwidth problem is

provided by Conjecture (2.4). Therefore, we decided to approach the cutwidth problem presented in this paper by looking for a good lower bound of the cyclic cutwidth of the n-dimensional cube.

3 FINDING THE LOWER BOUND FOR THE CYCLIC CUTWIDTH OF THE N-CUBE

The main goal of this project is to find a good lower bound for the cyclic cutwidth of the n-cube. As a means of accomplishing this goal, we began by looking at families of subgraphs of the n-dimensional cube for which we could calculate the cyclic cutwidth.

We started by looking at a 3-cube and figuring out how many edges we can take away before the cyclic cutwidth changed. Although this method would definitely give us a subgraph of the 3-cube with the same cyclic cutwidth, increasing the dimension would only complicate matters. Plus, there was an inconsistency and no apparent pattern to the number of edges removed when we used the same method on the 4 and 5-cubes. Therefore, we decided to disregard this method and try a different approach.

Our next approach to finding a subgraph was to build up to an n-cube instead of taking away from it. We knew that in order for a subgraph to represent a reasonable lower bound, it would have to contain a majority of the edges of the n-cube. We started by considering a 4-cube and a particular family of subgraphs that gives interesting results.

3.1 Subgraph of the 4-cube

If we take one edge in a 4-cube and find all 2-cubes which share that one particular edge, we will find there are always three 2-cubes which share that edge. The subgraph consisting of the union of these 2-cubes is interesting. We will prove that the subgraph of the 4-cube consisting of the three 2-cubes sharing one common edge has a cyclic cutwidth of 3.

Following the B. James [6] approach, we observe that each 2-cube can either contain the center (and contribute a cutwidth of 1 to the graph) or not contain the center (and contribute a cutwidth of 2 to the graph). (Refer back to Figure 4.)

Theorem (3.11) *The subgraph of the* 4*-cube consisting of all* 2*-cubes sharing one common edge has a cyclic cutwidth of* 3.

Proof Since there are three 2-cubes to look at, we see that four cases arise. These cases deal with the containment of the center by the 2-cubes.

CASE I

Suppose all three 2-cubes contain the center. Each will contribute a cut of one to the total cut. Thus, the cutwidth is 3.

CASE II

Suppose two 2-cubes contain the center. This will automatically yield a cut of two in all directions. When we place the remaining 2-cube on the host graph so that it does not contain the center, it will have to double back over itself. This will contribute a cut of two to some region of the host graph. Therefore, the host graph

will have a cut of four.

CASE III

Suppose one 2-cube contains the center. This will automatically yield a cut of one in all directions. The second 2-cube can be placed on either side of the common edge. Since this 2-cube will have to double back on itself, contributing a cut of two; thus, this will automatically yield a total cut of at least three.

CASE IV

Suppose none of the 2-cubes contain the center. Then the 2-cubes will have to be placed on either side of the common edge. There are only two possible layouts. One possible layout is when two of the 2-cubes are placed on one side of the common edge and the remaining 2-cube is placed on the other side. Since each 2-cube will have to double back on itself, each will contribute a cut of two. Thus, the side with two 2-cubes will contribute a total cut of four. The other layout is where all three of the 2-cubes are placed on the same side of the common edge. Clearly, this will contribute a cut of at least three.

(End of proof).

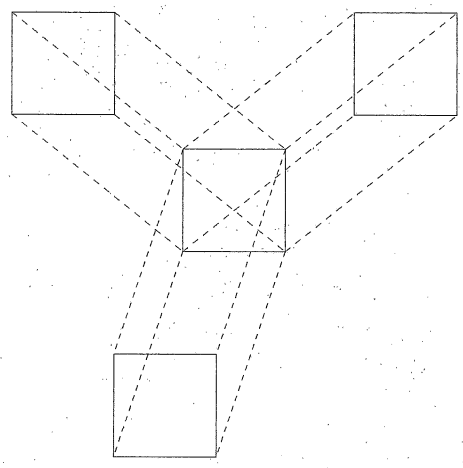
The cyclic cutwidth of the 4-cube is six [6]. Although this subgraph only gives us half of the cyclic cutwidth of the 4-cube, we notice that there is another disjoint copy of the same subgraph within the 4-cube. We also notice that the edge that shares all three 2-cubes in common has an "opposite edge" in the 4-cube. If u and v represent the vertices of one edge in the 4-cube, then there exists an unique "opposite edge." If we

label the vertex which is the furthest distance from vertex u as u' and the vertex which is the furthest distance from vertex v as v', then (u', v') will represent the "opposite edge." For example, if u is represented by $(0,0,1,0)$, then u' will be represented by $(1,1,0,1)$. Since the two copies of the subgraph are disjoint, we may be able to increase the cyclic cutwidth of our subgraph by laying both copies onto the same host graph. Before considering whether this is true, we decided to increase the dimension of our problem to see if we could find similar results or develop a pattern in higher dimensions.

3.2 Extending to higher dimensions

We next looked at a 5-dimensional cube. Our subgraph consists of taking one 2-cube and all 3-cubes sharing that one face. After drawing many examples, we were ready to prove that the cyclic cutwidth of our subgraph must be at least 5. But if we were to proceed in a manner similar to the previous proof, we would have twice the number of cases to examine.

After being able to provide a proof for the cyclic cutwidth of our subgraph for the 4-dimensional case, we wanted to be able to find the cyclic cutwidth of our subgraph for any dimension. We notice that our subgraph of a 5-dimensional cube consists of four 2-cubes arranged in the shape of a "*Y*". (See Figure 5.) The central 2-cube is connected to the three outer 2-cubes in a way to form three 3-cubes sharing a common face.



Y_5

Figure 5

We can generalize this subgraph of the cube for any dimension, n. The common $(n-3)$-cube will be arranged in the center of the "Y" and the three $(n-3)$-cubes will be arranged on the three endpoints of the "Y". The three $(n-3)$-cubes will have each of their vertices connected to each of the vertices of the common $(n-3)$-cube in a way to form cubes. Thus, when the common $(n-3)$-cube connects with the other $(n-3)$-cube, they will form an $(n-2)$-cube. We will denote this subgraph of the n-dimensional cube as Y_n.

3.3 Defining $\theta(S)$

A technique from [1] is useful here. In that paper, $\theta(S)$ is defined when $S \subseteq V$, where V is the vertex set of a given graph:

$$\theta(S) = |\{(v,w) \in E : v \in S, w \notin S\}|.$$

Basically, $\theta(S)$ counts the number of edges in G with one vertex in S and the other vertex not in S.

We let $\eta : V \to \{1,2,....,m\}$ be a numbering of the vertices of a graph, G. Then for each l, $0 \leq l \leq |V|$, define $S_l(\eta) = \eta^{-1}(\{1,2,...,l\})$. We know $lcw(G : \eta)$ can be thought of as the maximum number of edges that pass between any two vertices on the linear host graph. Then we know

$$lcw(G : \eta) = \max_{0 \leq \ell \leq m} \theta(S_\ell(\eta)).$$

But since we know that for the graph of the n-cube, the numbering scheme given by the lexicographic ordering of the vertices minimizes $\theta(S_\ell)$ for each ℓ,

$$lcw(Q_n) = \max_{0 \leq \ell \leq 2^n} \theta(S_\ell(lex)).$$

We also note that, from [1], we have two observations about $\theta(S)$:

(i) For all $S \subseteq V, \theta(S) = \theta(S^c)$. Thus

$$\min_{|S|=\ell} \theta(S) = \min_{|S|=m-\ell} \theta(S).$$

(ii) On Q_n, we have the following recursion.

$$\theta(S_{\ell,n}(lex)) = \begin{cases} 2\ell + \theta(S_{\ell,n-2}(lex)) & \text{if } 0 \leq \ell \leq 2^{n-2} \quad (a) \\ 2^{n-1} + \theta(S_{\ell-2^{n-2},n-2}(lex)) & \text{if } 2^{n-2} \leq \ell \leq 2^{n-1}. \quad (b) \end{cases}$$

3.4 Y-coloring scheme

Notice that in (ii) a recursion occurs. This gives some hope that we might be able

to solve the linear cutwidth problem for Y_n using a recurrence relation.

Recall that Y_n consists of three $(n-3)$-cubes, each with 2^{n-3} vertices, where each of the vertices are connected to the vertices of a central $(n-3)$-cube in a way to form three $(n-2)$-cubes sharing the central cube as a common face. (Refer back to Figure 5.) We "color" in the vertices to show they are included in the set S, and calculate $\theta(S)$ for each set S. Each vertex that we color and add to S will change both the internal and external cuts. The internal cut is the number of edges that lie within the $(n-3)$-cube and have one vertex in S and one vertex not in S. The external cut is the number of edges that connect two $(n-3)$-cubes and have one vertex in S and one vertex not in S. The marginal internal and marginal external cuts are the changes in the cuts by adding one new vertex to S.

There are several unique characteristics to the Y-coloring scheme. The first is that once one vertex from an $(n-3)$-cube is colored, we must keep coloring the vertices from the same $(n-3)$-cube until all of its vertices are colored. Only after all of the vertices of the $(n-3)$-cube are in S can we choose to start coloring the vertices of another $(n-3)$-cube.

The reasoning behind this is simple. Since the goal is to minimize the total cut every time a vertex is added to S, it would not make sense to add vertices from another $(n-3)$-cube to S. When a vertex from the original $(n-3)$-cube is added to S, the external cut will increase by one, and the internal cut will increase by less than $(n-3)$. If the vertex from another $(n-3)$-cube is added to S before all of the vertices from the

original $(n-3)$-cube are colored, then the external cut will still increase by one, but the internal cut will increase by $(n-3)$. Therefore, to minimize the total internal cut, it is clear that all of the vertices from the original $(n-3)$-cube should be colored before moving on to another $(n-3)$-cube.

A second characteristic is that the first vertex chosen in the Y-coloring scheme is never from the central $(n-3)$-cube. The reasoning is quite clear. If a vertex is chosen from the central $(n-3)$-cube, then we would have an internal cut of $(n-3)$, and an external cut of 3 (one going to each of the outer $(n-3)$-cubes.) But, if we choose a vertex from one of the outer $(n-3)$-cubes, we still have the same internal cut, but an external cut of one. Since the Y-coloring scheme aims to minimize the total cut, it is best to start by choosing one of the vertices from one of the outer $(n-3)$-cubes.

Another characteristic of the Y-coloring scheme is that the maximum θ occurs when one $(n-3)$-cube is completely colored and a second $(n-3)$-cube is partially colored. This statement is proven by reducing three out of the four cases down to the fourth case.

First, suppose only one of the outer $(n-3)$-cubes has a portion, say K^* (where $K^* < 2^{n-3}$), of its vertices colored. Then $\theta(S_{K^*})$ will consist of both an internal and external cut, where the external cut will be K^*. But if we follow the numbering characteristic of the Y-coloring scheme and finish coloring the vertices of the first $(n-3)$-cube and start coloring the vertices of a second $(n-3)$-cube, it is clear that $\theta(S_{2^{n-3}+K^*})$ will consist of the same internal cut, but a total external cut equal to

$2^{n-3} + K^*$. Since $\theta(S_{K^*}) < \theta(S_{2^{n-3}+K^*})$, θ is not optimized when only a portion of the vertices of one $(n-3)$-cube are colored in.

Second, suppose three of the $(n-3)$-cubes have all of their vertices colored and K^* vertices from the remaining $(n-3)$-cube are colored. Since $\theta(S_{2^{n-3}-K^*}) = \theta\left(S_{3(2^{n-3})+K^*}\right)$ by observation (i), this configuration is the complement of the first case. Since $\theta(S_{2^{n-3}-K^*}) < \theta\left(S_{2^{n-3}+(2^{n-3}-K^*)}\right)$, then $\theta\left(S_{3(2^{n-3})+K^*}\right) < \theta\left(S_{2^{n-3}+(2^{n-3}-K^*)}\right)$. Therefore, θ is not optimized when the vertices of three of the $(n-3)$-cubes are colored in.

The remaining case is when all of the vertices from two $(n-3)$-cubes and K^* of the vertices from the central $(n-3)$-cube are colored. $\theta\left(S_{2(2^{n-3})+K^*}\right)$ will consist of an internal cut, plus an external cut of $2(2^{n-3}) + K^*$. By observation (i), we see that $\theta\left(S_{2(2^{n-3})+K^*}\right) = \theta\left(S_{2^{n-3}+(2^{n-3}-K^*)}\right)$. Therefore, we choose $\theta\left(S_{2^{n-3}+(2^{n-3}-K^*)}\right)$.

A final characteristic of the Y-coloring scheme is that θ is maximized when at least half of the vertices are colored in the second colored $(n-3)$-cube. Let K^* represent the number of vertices colored in a partially colored $(n-3)$-cube. By observation (i), the internal contribution of a set of vertices of size K^* is equal to the internal contribution of a set of vertices of size $2^{n-3} - K^*$. If $K^* < \frac{1}{2}(2^{n-3})$, then $2^{n-3} - K^* \geq \frac{1}{2}(2^{n-3})$. The external contribution of a set of vertices of size K^* will be less than $\frac{1}{2}(2^{n-3})$. But the external contribution of a set of vertices of size $2^{n-3} - K^*$ will be at least $\frac{1}{2}(2^{n-3})$. Therefore, to maximize the external cut of a set of vertices of size K, we would want to choose the number of colored vertices to always be at least

$\frac{1}{2}(2^{n-3})$.

From these observations, we see that θ is maximized when one of the $(n-3)$-cubes has all of its vertices completely colored and a second $(n-3)$-cube has at least half of its vertices colored.

Lemma (3.41) *The solution of the problem of maximizing $\theta(Y_{K,n})$ with respect to the Y-coloring scheme is given by the following recurrence relation*:

$$\theta(Y_{K,n}) = 7(2^{n-5}) + \theta\left(Y_{9(2^{n-5})-K,\, n-2}\right).$$

Proof Let K represent the total number of colored vertices in Y_n. Then by previous discussion, $K = 2^{n-3} + K^*$ (where K^* is some number of colored vertices of the second cube). Either $\frac{3}{4}(2^{n-3}) \le K^* \le (2^{n-3})$ or $\frac{1}{2}(2^{n-3}) \le K^* \le \frac{3}{4}(2^{n-3})$. Begin by assuming $\frac{3}{4}(2^{n-3}) \le K^* \le (2^{n-3})$.

$$\theta(Y_{K,n}) = K + \theta(Q_{K^*,n-3})$$

$$= K + \theta(Q_{2^{n-3}-K^*,n-3}) \text{ by observation (i).}$$

By using the formula from observation (ii) part (a), with $\ell = 2^{n-3} - K^*$, we see

$$\theta(Y_{K,n}) = K + 2(2^{n-3} - K^*) + \theta(Q_{2^{n-3}-K^*,n-5})$$

$$= K + 2(2^{n-2} - K) + \theta(Q_{2^{n-2}-K,n-5})$$

$$= 2^{n-1} - K + \theta(Q_{2^{n-2}-K,n-5}).$$

We know that for $2^{n-5} \le m \le 2(2^{n-5})$,

$$\theta(Y_{m,n-2}) = m + \theta(Q_{m-2^{n-5},n-5}),$$

$$\Rightarrow \theta(Q_{m-2^{n-5},n-5}) = \theta(Y_{m,n-2}) - m.$$

Let $m - 2^{n-5} = 2^{n-2} - K$.

Thus, $m = 9(2^{n-5}) - K$.

Then,

$$\begin{aligned}\theta(Y_{K,n}) &= 2^{n-1} - K + \theta(Q_{2^{n-2}-K,n-5}) \\ &= 2^{n-1} - K + \theta(Q_{m-2^{n-5},n-5}) \\ &= 2^{n-1} - K + \theta(Y_{m,n-2}) - m \\ &= 2^{n-1} - K + \theta\left(Y_{9(2^{n-5})-K,n-2}\right) - 9(2^{n-5}) + K \\ &= 16(2^{n-5}) + \theta\left(Y_{9(2^{n-5})-K,n-2}\right) - 9(2^{n-5}) \\ &= 7(2^{n-5}) + \theta\left(Y_{9(2^{n-5})-K,n-2}\right).\end{aligned}$$

Thus, when $\frac{3}{4}(2^{n-3}) \le K^* \le (2^{n-3})$, we have the following recurrence relation:

$\theta(Y_{K,n}) = 7(2^{n-5}) + \theta\left(Y_{9(2^{n-5})-K,n-2}\right)$.

The proof of the other case follows very similarly. Assume $K = 2^{n-3} + K^*$ where $\frac{1}{2}(2^{n-3}) \le K^* \le \frac{3}{4}(2^{n-3})$.

Then, $\theta(Y_{K,n}) = K + \theta(Q_{K^*,n-3})$

$= K + \theta(Q_{2^{n-3}-K^*,n-3})$ by observation (i).

By using the formula from observation (ii) part (b), with $\ell = 2^{n-3} - K^*$, we see

$$\begin{aligned}\theta(Y_{K,n}) &= K + 2^{n-4} + \theta(Q_{2^{n-3}-K^*-2^{n-5},n-5}) \\ &= K + 2^{n-4} + \theta\left(Q_{2^{n-5}-[2^{n-3}-K^*-2^{n-5}],n-5}\right).\end{aligned}$$

We can simplify $2^{n-5} - 2^{n-3} + K^* + 2^{n-5}$ to $2^{n-4} + K^* - 2^{n-3}$.

But since $K^* = K - 2^{n-3}$,

$$\begin{aligned}2^{n-4} + K^* - 2^{n-3} &= 2^{n-4} + K - 2^{n-3} - 2^{n-3}, \\ &= 2^{n-4} + K - 2^{n-2}.\end{aligned}$$

Therefore, $\theta(Y_{K,n}) = K + 2^{n-4} + \theta(Q_{K+2^{n-4}-2^{n-2},n-5})$.

We know that for $2^{n-5} \leq m \leq 2(2^{n-5})$,

$$\theta(Y_{m,n-2}) = m + \theta(Q_{m-2^{n-5},n-5})$$

$$\Rightarrow \theta(Q_{m-2^{n-5},n-5}) = \theta(Y_{m,n-2}) - m.$$

Let $m - 2^{n-5} = K + 2^{n-4} - 2^{n-2}$ and solve for m.

Thus, $m = 2^{n-5} + K + 2^{n-4} - 2^{n-2}$

$$= 2^{n-5} + K + 2(2^{n-5}) - 8(2^{n-5})$$

$$= K - 5(2^{n-5}).$$

Then $\theta(Y_{K,n}) = K + 2^{n-4} + \theta(Q_{K+2^{n-4}-2^{n-2},n-5})$

$$= K + 2^{n-4} + \theta(Q_{m-2^{n-5},n-5})$$

$$= K + 2^{n-4} + \theta(Y_{m,n-2}) - m$$

$$= K + 2^{n-4} + \theta\left(Y_{K-5(2^{n-5}),n-2}\right) - [K - 5(2^{n-5})]$$

$$= 7(2^{n-5}) + \theta\left(Y_{K-5(2^{n-5}),n-2}\right).$$

Thus, when $\frac{1}{2}(2^{n-3}) \leq K^* \leq \frac{3}{4}(2^{n-3})$, we have the following recurrence relation: $\theta(Y_{K,n}) = 7(2^{n-5}) + \theta\left(Y_{K-5(2^{n-5}),n-2}\right)$.

Comparing the two recurrence relations just obtained, we observe an apparent discrepancy. However, since $K - 5(2^{n-5})$ and $9(2^{n-5}) - K$ are complementary (that is, sum to 2^{n-3}), by observation (i), the two recurrence relations are identical.

(End of proof).

3.5 Solving the recurrence relation

Now that we have the recurrence relation $A_n = 7(2^{n-5}) + A_{n-2}$, with its two initial conditions ($A_3 = 2$ and $A_4 = 4$), we must solve this non-homogenous relation. We start by solving the homogenous part, which gives $A_n = C_1(1)^n + C_2(-1)^n$. Then we calculate the non-homogenous part and add its result to the homogenous solution, which gives $A_n = C_1 + C_2(-1)^n + \frac{7}{24}(2^n)$. Using the initial conditions to solve for C_1 and C_2, we get $C_1 = -\frac{1}{2}$ and $C_2 = -\frac{1}{6}$ which gives us the following lemma.

Lemma (3.51) $lcw(Y_n) = -\frac{1}{2} - \frac{1}{6}(-1)^n + \frac{7}{24}(2)^n$.

Proof $lcw(Y_n)$ is the minimum over all numberings of the maximum values of θ. But since the Y-coloring scheme is the optimal numbering, its maximum, given by the solution to the recurrence relation, is $lcw(Y_n)$.

(End of proof).

3.6 Relationships between *lcw*(*G*) and *ccw*(*G*)

We also have another useful lemma. Let G be a graph embedded onto a cyclic host, H. Let C_{Max} be the maximum of the cuts between neighboring vertices of H and, C_{Min} be the minimum of the cuts between neighboring vertices of H.

Lemma (3.61) *The sum of* C_{Max} *and* C_{Min} *will be at least* $lcw(G)$. *That is,* $C_{Max} + C_{Min} \geq lcw(G)$.

Proof Assume that $C_{Max} + C_{Min} < lcw(G)$. If we take all of the edges that contribute to C_{Min} and place them on the opposite side of the center, we would decrease the cut

of C_{Min} to 0; thus, creating a linear layout of the vertices of G around the center. Since $lcw(G)$ is the minimum over all numberings, $lcw(G) \leq C_{Max} + C_{Min}$. We have a contradiction since,

$lcw(G) \leq C_{Max} + C_{Min} < lcw(G)$.

Thus, $C_{Max} + C_{Min} \geq lcw(G)$.

(End of proof).

We use this lemma to support the following theorem. It shows how the cyclic cutwidth of two disjoint subgraphs are related to the linear cutwidth of one of the disjoint subgraphs.

Theorem (3.62) *Let G_1 and G_2 be two isomorphic graphs, and let $G_1 \cup G_2$ be the disjoint union of G_1 and G_2. Then $ccw(G_1 \cup G_2) = lcw(G_2)$.*

Proof Let H_1 and H_2 be the cyclic hosts for G_1 and G_2, respectively. Let $C_{1_{Max}}$ be the maximum of the cuts between neighboring vertices of H_1 and $C_{1_{Min}}$ be the minimum of the cuts between neighboring vertices of H_1. Also, let $C_{2_{Max}}$ be the maximum of the cuts between neighboring vertices of H_2 and $C_{2_{Min}}$ be the minimum of the cuts between neighboring vertices of H_2. Then by Lemma 3.61,

$C_{1_{Max}} + C_{1_{Min}} \geq lcw(G_1)$ and $C_{2_{Max}} + C_{2_{Min}} \geq lcw(G_2)$. Also,

$ccw(G_1 \cup G_2) \geq \max(C_{1_{Max}} + C_{2_{Min}}; C_{2_{Max}} + C_{1_{Min}})$.

Assume without the loss of generality, $C_{1_{Max}} \geq C_{2_{Max}}$.

Adding $C_{2_{Min}}$ to both sides, we get $C_{1_{Max}} + C_{2_{Min}} \geq C_{2_{Max}} + C_{2_{Min}} \geq lcw(G_2)$.

Therefore, $ccw(G_1 \cup G_2) \geq lcw(G_2)$.

To show the reverse inequality, label the vertices of G_1 by $v_1, v_2, ..., v_n$, and the vertices of G_2 by $u_1, u_2, ..., u_n$. Then arrange the vertices of $(G_1 \cup G_2)$ in the cyclic layout so v_1 is adjacent to v_2, which is adjacent to v_3, and so on to v_n, and v_n is adjacent to the u_1, which is adjacent to u_2, which is adjacent to u_3, and so on to u_n, which is adjacent to v_1. Clearly, we see that the $ccw(G_1 \cup G_2) \leq lcw(G_2)$.

Thus, $ccw(G_1 \cup G_2) = lcw(G_2)$.

(End of proof).

From this theorem, we know that we can use the linear cutwidth of Y_n to find the cyclic cutwidth of the union of two copies of Y_n. This would provide a lower bound of the cyclic cutwidth that we are looking for.

Knowing the solution to our recurrence relation, we have following theorem.

Theorem (3.63) $ccw(Y_n \cup Y_n) = -\frac{1}{2} - \frac{1}{6}(-1)^n + \frac{7}{24}(2)^n$.

Proof The proof of this theorem follows clearly from Lemma (3.51) and Theorem (3.62).

3.7 Other discoveries from our research

We can use Theorem (3.62) which relates the linear cutwidth of a graph, G, to the cyclic cutwidth of the union of two copies of G, $G \cup G$, to obtain a better lower bound for the cyclic cutwidth problem. In the n-cube, there are disjoint copies of two $(n-1)$-cubes as subgraphs. We know that the $(n-1)$-cube is a more complete subgraph than our Y_n. Thus, we obtain the following improved lower bound.

Theorem (3.71) $ccw(Q_n) \geq lcw(Q_{n-1}) = \begin{cases} \frac{2^n-1}{3} & \text{if } n \text{ is even} \\ \frac{2^n-2}{3} & \text{if } n \text{ is odd.} \end{cases}$

Proof The formula for the $lcw(Q_n)$ is given in [1]. The n-dimensional cube contains disjoint copies of two $(n-1)$ cubes. Letting $G_1 = Q_{n-1}$ and $G_2 = Q_{n-1}$, and using Theorem (3.62), we can calculate an improved lower bound for the cyclic cutwidth.

(End of proof).

4 CONCLUSION

4.1 Results

Although the Chavez-Trapp Conjecture (2.4) has yet to be verified for all cases, we found a lower bound that can be used to give a good estimate for the cyclic cutwidth problem. By looking at a family of subgraphs of the n-cube, which we called Y_n, we were able to find two disjoint copies of the same subgraph in the n-cube. Then using Theorem (3.62), we were able to state that the cyclic cutwidth of two disjoint copies of Y_n is equal to the linear cutwidth of one copy of Y_n. Although a formula for finding the linear cutwidth of Y_n did not exist, we were able to use the definition of $\theta(S)$ and the "Y-coloring" scheme to find the linear cutwidth of Y_n.

Calculating $\theta(S)$ for Y_n where n is small was not difficult. But as n increased, so did the complexity of calculating $\theta(S)$. Therefore, using a technique found in [1], we were able to find a recurrence relation for calculating $lcw(Y_n)$. This proved to be an important step towards solving our lower bound problem.

After all of the results were calculated, we found a better lower bound using two disjoint $(n-1)$-cubes instead of Y_n as our subgraph. Using Theorem (3.71), we were able to find a lower bound that is approximately 80% of the upper bound given by Conjecture (2.4). This discovery was uncovered after realizing that every n-dimensional cube contains two disjoint $(n-1)$-cubes as subgraphs. Then applying Theorem (3.62), we were able to prove Theorem (3.71).

4.2 Future research

Since a lower bound was found, the immediate question for future research is "what keeps us from solving the cyclic cutwidth problem for the n-dimensional cube?" It would be nice to improve our lower bound to get results closer to the results from Conjecture (2.4).

The subgraph composed of the disjoint union of two $(n-1)$-cubes is missing exactly 2^{n-1} edges from the complete n-dimensional cube. We could spend some time investigating how adding these "missing edges" might increase the lower bound in Theorem (3.71).

Another area of interest might be to use Theorem (3.62) to solve other applications or problems. This theorem might be applied to simplify the proofs of difficult problems.

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